Oil Price Dynamics (2002-2006)

by

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Abstract

Recent crude oil price dynamics have been characterized by high volatility, high intensity jumps, and strong upward drift, indicating that oil markets have been out-of-equilibrium. In this paper, an explanation of the oil price process based on oil market and world economic fundamentals is provided, with the indication that pressure on oil prices have resulted mainly from rigid crude oil supply and an expanding world demand for crude oil. A change in the oil price process parameters would require a change in the underlying fundamentals. Market expectations, extracted from call and put option prices, anticipated in the short term no change in the underlying fundamentals. Markets expected oil prices to remain volatile and jumpy, and with higher probabilities to rise, rather than fall, above the expected mean.

Classification: C1 C4 G1 Q4

Keywords: Oil prices, option pricing, jump-diffusion, variance-gamma distribution, Fourier space

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I. Introduction

Under the combination of rapidly expanding world demand for crude oil and tight world crude supply, crude oil prices have jumped in recent years. By breaking a record level of US$78.30 per barrel (bl) on August 7, 2006[1] and remaining comfortably in the neighborhood of US$75/bl during much of 2006, crude oil prices had risen to uncharted levels. While economic agents, including traders, investors, speculators, and policymakers were following developments in crude oil prices closely, not much was known about the stochastic processes driving these prices. Contrary to stock market indices, for which an abundant and advanced modeling literature now exists, crude oil prices, in spite of their importance, have not been the subject of extensive modeling research. Knowledge of their underlying stochastic process is highly relevant not only for pricing derivatives and hedging, but also for policymaking and short-term forecasting.

In this paper we address the dynamics of daily oil prices during the period from January 2, 2002 to July 7, 2006.[2] At the outset numerous striking facts regarding oil markets should be stated. Foremost, global demand pressure for oil kept increasing during this period, causing oil prices to rise by more than threefold, from US$21.13/bl on January 2, 2002 to US$73.76/bl on July 7, 2006.[3] Second, the noted ascent in oil prices was not monotonic or smooth; oil prices rose, often to a new record, retreated temporarily, then resumed their move to higher record; their movements were dominated by high intensity jumps, indicating that oil markets were constantly out-of-equilibrium. Third, oil price volatilities were excessively high. As measured by the implied volatility, volatility was in the range of 30 percent, implying that oil markets were facing big uncertainties regarding future price developments and were sensitive to small shocks and to news. Finally, market expectations, extracted from crude oil call and put option prices, were right-skewed. More specifically, markets held higher probabilities for further price increases than price decreases. Moreover, markets seemed to expect large upward jumps in oil prices, as reflected by the price and volume of options at strikes in the range of US$75–US$85/bl.

The paper is structured as follows. In Section II, we describe the time series properties of oil prices and the empirical distribution of oil price returns. In Section III, we model oil prices as a Merton (1976) jump-diffusion (J-D) process. In Section IV, oil price returns are modeled as a Levy process of the variance-gamma type (Madan and Milne, 1991; and Madan et al.,

[1] When British Petroleum shut down the Prudhoe Bay field in Alaska for pipeline maintenance.

[2] Futures' contracts on Brent, three-month delivery; the sample contains 1130 observations. The source is Reuters.

[3] The recessionary effect of high oil prices has been studied by Hamilton (1983). A considerable literature thereafter has dealt with the relationship between oil shocks and real GDP. By causing a general increase in the price level, oil shocks, ceteris paribus, reduces real cash balances and therefore aggregate demand.

1998). In Section V, we discuss option pricing in the Fourier space (Heston 1993, and Carr and Madan, 1999). In Section VI, we present an oil price density forecast based on option prices. Our conclusions are in Section VII.

II. EMPIRICAL ASPECTS OF FUTURES OIL PRICES DURING 2002–2006

A. Recent trends and descriptive statistics

With a view to concentrating on recent oil prices dynamics, the chosen sample period was January 2, 2002–July 7, 2006, containing 1,130 daily observations. In Figure 1 we illustrate the daily behavior of oil prices. It clearly shows that oil prices were moving upward, and have become forecastable. After each peak, oil prices seemed to retreat temporarily then re-trended toward higher peaks. Let S_t be the futures price in US$/bl. An augmented Dickey-Fuller test (Table 1) indicated that S_t possessed a unit root; it was pulled by an upward trend, showing no sign for mean reversion. Changes in S_t, defined as $\Delta S_t = S_t - S_{t-1}$, were, however, stationary. Based on the unit-root test, the dynamics of the oil process were represented by a simple auto-regression of order two (AR2) which yielded good fit and highly significant coefficients, namely:

$$S_t = \underset{t=31.3}{0.93} S_{t-1} + \underset{t=2.3}{0.07} S_{t-2} + \underset{t=0.98}{0.09}, \ R^2 = 0.99, \text{ DW}=2.05 \quad (1)$$

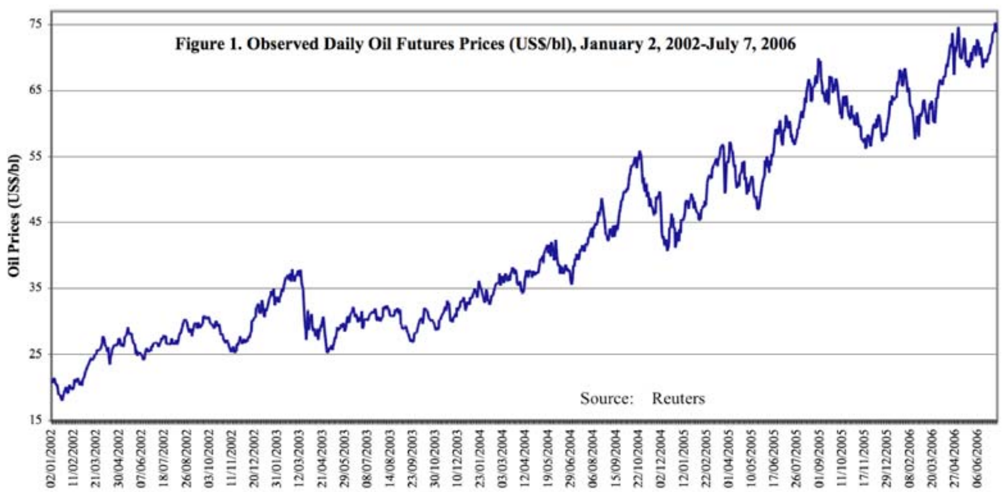


Figure 1. Observed Daily Oil Futures Prices (US$/bl), January 2, 2002-July 7, 2006

Table 1. Time-Series Properties of Oil Prices

Augmented Dickey-Fuller Unit-Root Test on Oil Prices.
Null Hypothesis: S_t has a unit root. Augmented Dickey-Fuller test statistic=-0.52; probability value=0.88. Test critical values: 1% level (-3.44); 5% level (-2.86); 10% level (-2.57).
Null Hypothesis: $\Delta S_t = S_t - S_{t-1}$ has a unit root. Augmented Dickey-Fuller test statistic=-35.98; probability value=0.00. Test critical values: 1% level (-3.44); 5% level (-2.86); 10% level (-2.57).

These price dynamics reflected the underlying fundamentals of the oil markets.[4] In spite of rising prices, world crude oil supply was rigid at about 81 million barrels per day (mbd) for most of the sample period.[5] World crude oil demand was, however, strongly stimulated by a world economy growing at 4–5 percent per year during 2002–06, excessively low nominal interest rates, and sharply depreciated US dollar.[6] Additionally, world crude oil demand is known to be highly price inelastic in the short term. Price elasticity of demand ranges between -0.01 and -0.04. More specifically, significant increase in oil price would have only a small negative impact on oil demand. World crude oil demand is also known to have high income elasticity. If the technical coefficient between crude oil and real GDP is fixed in the short tem, then income elasticity could be close to one. Econometric estimates, however, show that short-run income elasticity ranges between 0.2 and 0.4. Thus rigidity of crude supply, combined with an expanding world demand for crude oil, has resulted in growing demand-supply imbalances. Given price inelasticities of both oil demand and supply, even a small excess demand (supply) for oil would require large changes in oil prices to clear markets.

Figure 2. Daily Crude Oil Price Returns Distribution, Jan 2, 02-July 7, 06.

Descriptive statistics: mean = 0.116; standard deviation=2.29; skewness=-0.39; kurtosis=4.79; Jarque-Bera normality statistics=179.3, probability-value=0.0.

Additional insight into oil price dynamics is uncovered by analyzing the log-price return defined as $x_t = \Delta \log S_t = \log S_t - \log S_{t-1}$. The graph for these changes (Figure 2) shows that large jumps in crude oil prices were frequent and had a relatively high probability. Although the mode was around 1–2 percent, daily changes in the range of 5–7 percent were not uncommon.[7] The empirical distribution had a large dispersion, with standard deviation

[4] Investors and speculators, through opening and closing positions on the futures markets, affect price dynamics and increase price volatility. However, their role is limited to the short run. Given the sample period under study, underlying fundamentals were key determinants of the oil price process. Incidentally, the IMF World Economic Outlook, September 2006, could not establish evidence for a long-term effect of speculation on oil prices.

[5] See, for instance, The International Energy Agency, Oil Market Report, September 2006.

[6] World economy was reported to have grown at about 4-5 percent in real terms during 2002-2006. See International Monetary Fund, World Economic Outlook, September, 2006.

[7] The frequency of jumps exceeding ± 3 percent was estimated from the sample at 23 percent.

estimated at 2.29 (annualized to 36.3 percent). The distribution was left-skewed, implying that downward jumps of smaller size were more frequent than upward jumps of larger size; as the mean was positive and high, smaller jumps were outweighed by larger jumps. The distribution had also fat tails, meaning that large jumps tended to occur more frequently than in the normal case. These empirical findings on daily oil futures prices were typical of financial time series as noted in Clark (1973), Fama (1965), and Mandelbrot (1963). These facts suggested modeling the oil price process as a jump-diffusion or, in a more general way, as a Levy process (Cont and Tankov, 2004).

B. Oil Price Time-Varying Volatility

Volatility measures uncertainty and also sensitivity of prices to news and shocks, and is a key parameter in option pricing. Two types of volatilities are studied here: the implied volatility from crude oil call options,[8] and volatility computed by a GARCH (1, 1) model. Both measures point to high volatility in futures prices.

Data on implied volatility for Brent futures options for August 2005–June 2006 indicated that oil price volatility was high (Figure 3). While averaging 30 percent, volatility often surged to 34–35 percent, indicating that oil markets were experiencing big uncertainty regarding expected price developments, and were highly sensitive to small shocks and news. Volatility pattern shows volatility clustering with rising pressure on oil prices and volatility decline with reduced pressure on oil prices. High volatility increases speculative demand for futures contracts, which in turn leads to higher volatility and volatility clustering.

Volatility was also computed using a GARCH model for data on daily oil futures prices covering January 2, 2002–July 7, 2006 (Figure 4). The oil price return was defined as: $x_t = \Delta \log S_t = \log S_t - \log S_{t-1}$.[9] The fitting of the GARCH model showed high price volatility, periods of volatility clustering, followed by some reversion to a mean volatility estimated at 43 percent. GARCH volatility was rising during periods of large price shocks, stimulating speculation and leading to volatility clustering; it was, however, receding during periods of price retreat. It corroborated the observed implied volatility, namely oil markets were constantly experiencing large uncertainties and were affected by frequent shocks.

[8] Implied volatility is the volatility which equates the Black-Scholes (1973) call option pricing formula with the call option’s market value.

[9] GARCH stands for Generalized Autoregressive Conditional Heteroskedasticity. A GARCH (1,1) model is defined as follows: The mean equation: $x_t = \Delta \log S_t = c + \varepsilon_t$, $\varepsilon_t \sim N(0, \sigma_t^2)$

The conditional variance equation: $\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$, where $\sigma_t^2 = E(\varepsilon_t^2)$

Figure 3. Crude Oil Futures Implied Volatility, August 05- June 06.

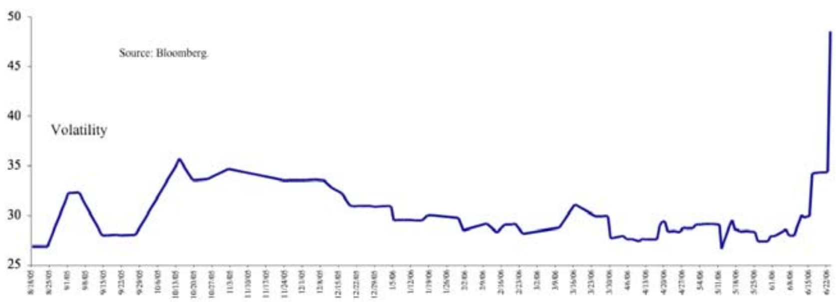
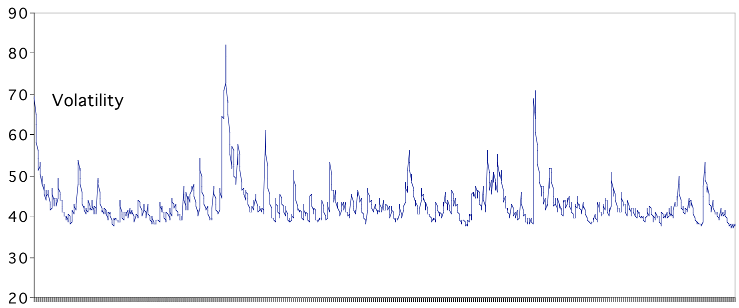


Figure 4. Estimated GARCH (1,1) Volatility, Jan 02-July 06



III. Crude oil price as a Merton Jump-Diffusion Process

A. The Stochastic Differential Equation for the Jump-Diffusion Model

Based on the empirical findings of the previous section, namely the presence of skewness and kurtosis in the empirical distribution of oil price returns, an adequate model for oil prices would be a jump-diffusion model. In fact, Merton (1976), recognizing the presence of jumps in asset prices, for more accurate option pricing proposed modeling these prices as a jump-diffusion process instead of a pure diffusion model. Moreover, it is well-known that short-term options have market implied volatilities that exhibit a significant skew across strikes. In this connection, Bakshi et al. (1997) argued that pure diffusion based models have difficulties explaining the smile effect in short-dated option prices and emphasized the importance of adding a jump component in modeling asset price dynamics. In the same vein, Bates (1996) noted that diffusion-based stochastic volatility models could not explain skewness in implied volatilities, except under implausible values for the model's parameters. Models with jumps generically lead to significant skews for short-term maturities. More generally, adding jumps to returns in a diffusion-based stochastic volatility model, the resulting model can generate sufficient variability and asymmetry in the short-term returns to match implied volatility skews for short-term maturities.

Accordingly, the continuous-time stochastic process driving crude oil prices can be stated as a J-D process given by a stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dB_t + \left(\exp\left(J_t\right) - 1\right) dN_t \qquad (2)$$

S_t denotes crude oil price, α is the instantaneous return, and σ^2 is the instantaneous variance. The continuous component is given by a standard Brownian motion, B_t, distributed as $dB_t \sim N(0, dt)$. The discontinuities of the price process are described by a Poisson counter N_t, characterized by its intensity, λ, and jump size, J_t. The Brownian motion and the Poisson process are independent. The intensity of the Poisson process describes the mean number of arrivals of abnormal information per unit of time and is expressed as: $\mathrm{Pr}ob\left[\Delta N_t = 1\right] = \lambda dt$, and $\mathrm{Pr}ob\left[\Delta N_t = 0\right] = 1 - \lambda dt$. When abnormal information arrives, crude oil price jumps from S_{t-} (limit from left) to $S_t = \exp\left(J_t\right) S_{t-}$. The percentage change is measured by $\left(\exp\left(J_t\right) - 1\right)$. The jump size, J_t, is independent of B_t and N_t, and is assumed to be normally distributed: $J_t \sim N(\beta, \delta^2)$. Letting $X_t = \log\left(S_t\right)$ and using Ito's lemma, the log price return process becomes:

$$dX_t = \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t + J_t dN_t = \mu dt + \sigma dB_t + J_t dN_t \qquad (3)$$

where $\mu = \left(\alpha - \frac{1}{2}\sigma^2\right)$.[10] The parameter vector associated with the price process is therefore $\theta = \left(\mu, \sigma^2, \lambda, \beta, \delta^2\right)$. Discretized over $\left(t, t + \Delta\right)$, the model takes the form:

$$\Delta X_t = \mu\Delta + \sigma\Delta B_t + \sum_{i=0}^{\Delta N_t} J_i \qquad (4)$$

Where $\Delta B_t = B_{t+\Delta} - B_t \sim N\left(0, \Delta\right)$, and $\Delta N_t = N_{t+\Delta} - N_t$ is the actual number of jumps occurring during the time interval $\left(t, t + \Delta\right)$, and J_i are independently and identically distributed as $J_i \sim N(\beta, \delta^2)$. The log-return, $x_t = \Delta X_t$, includes therefore the sum of two independent components: a diffusion component with drift and a jump component. Its probability density is a convolution of two independent random variables and can be expressed as:[11]

[10] A solution to this SDE can be written as $S_T = S\left(0\right) \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right) T + \sigma B_T + \sum_{i=0}^{N_T} J_i\right]$

[11] Ball and Torous (1985) modeled the jump component in Merton's model as a Bernoulli process. In this respect, either one or no abnormal event occurs during the time interval $\left(t, t + \Delta\right)$, with Prob[one abnormal event]= $\lambda\Delta$, Prob[no abnormal event]=$1 - \lambda\Delta$, and Prob[more than one abnormal event]=0. The density function for the log-return becomes: $f\left(x\right) = N\left(\mu\Delta + n\beta, \sigma^2\Delta + n\delta^2\right)\left(\lambda\Delta\right) + N\left(\mu\Delta, \sigma^2\Delta\right)\left(1 - \lambda\Delta\right)$

$$f(x)=\sum_{i=0}^{\infty}\frac{(\lambda\Delta)^n e^{-\lambda\Delta}}{n!}\left[\frac{1}{\sqrt{2\pi\left(\sigma^2\Delta+n\delta^2\right)}}\exp\left(-\frac{\left(x-\mu\Delta-n\beta\right)^2}{2\left(\sigma^2\Delta+n\delta^2\right)}\right)\right] \quad (5)$$

With $n=0,1,2,.......$ Putting $\Delta=1$, i.e., the time interval is $(t,t+1)$, the density function becomes

$$f(x)=\sum_{i=0}^{\infty}\frac{(\lambda)^n e^{-\lambda}}{n!}\left[\frac{1}{\sqrt{2\pi\left(\sigma^2+n\delta^2\right)}}\exp\left(-\frac{\left(x-\mu-n\beta\right)^2}{2\left(\sigma^2+n\delta^2\right)}\right)\right] \quad (6)$$

B. Alternative Methods for Estimating the Jump-Diffusion Model: Maximum Likelihood, Method of Cumulants, and Method of Characteristic Function

1. The maximum likelihood method: Let $x=\{x_1,x_2,.......,x_T\}$ be an observed sample of log returns, the log-likelihood function can be expressed as:

$$L(\theta;x)=-T\lambda-\frac{T}{2}\ln(2\pi)+\sum_{t=1}^{T}\ln\left[\sum_{j=0}^{\infty}\frac{\lambda^n}{n!}\frac{1}{\sqrt{\sigma^2+n\delta^2}}\exp\left(\frac{-\left(x_t-\mu-n\beta\right)^2}{2(\sigma^2+n\delta^2)}\right)\right] \quad (7)$$

Application of the maximum likelihood (ML) method for estimating the J-D model has met with difficulties arising mainly from the identification of the jump parameter and instability of parameter estimates. Nonetheless, Ball and Torous (1983) applied directly the ML method by truncating the number of jumps at $n=15$. Ball and Torous (1985) and Jorion (1988) applied the ML method by assuming a Bernoulli process for the jump component. While the ML estimates achieve the lower bound for Cramer-Rao efficiency criterion, difficulties with the likelihood function arising from computational tractability, un-boundedness over the parameter space, and instability of parameters, have led researchers to explore alternative estimation methods, based essentially on the method of moments.

2. The method of cumulants (See Annex): Press (1967) used the method of cumulants as described in Kendall and Stuart (1977) to estimate the J-D model. Define the characteristic function (CF) of X_t as:

$$\phi_X(u)=E\left[\exp\left(iuX_t\right)\right]=\int\exp\left(iuX_t\right)f\left(X_t\right)dX_t \quad (8)$$

where $f\left(X_t\right)$ is the probability density function of X_t, u is the transform variable, and $\sqrt{-1}=i$.[12] The cumulants of X_t, denoted by κ_n, $n=0,1,2,....$, are the coefficients in the power series expansion of the logarithm of the CF of X_t, expressed as:

[12] The characteristic function $\phi_X(u)$ is related to the moment generating function $G_X(u)$ $G_X(u)=E\left[\exp\left(uX_t\right)\right]=\int\exp\left(uX_t\right)dF\left(X_t\right)$ by a change of the transform variable $u\rightarrow -iu$, namely $G_X(iu)=\phi_X(u)$, and $G_X(u)=\phi_X(-iu)$.

$$\ln \phi(u) = \sum_{n=1}^{\infty} \kappa_n \frac{(iu)^n}{n!} = 1 + \kappa_1 \frac{(iu)}{1!} + \kappa_2 \frac{(iu)^2}{2!} + ... + \kappa_n \frac{(iu)^n}{n!} + ... \quad (9)$$

Noting that the CF for the jump-diffusion process is given by:[13]

$$\phi_{X_1}(u) = \exp\left[-\frac{\sigma^2 u^2}{2} + i\mu u + \lambda\left(\exp\left(i\beta u - \frac{\delta^2 u^2}{2}\right) - 1\right)\right] \quad (10)$$

It follows that the first four cumulants of the J-D process are:

$$\kappa_1 = \mu + \lambda\beta,\ \kappa_2 = \sigma^2 + \lambda\delta^2 + \lambda\beta^2,\ \kappa_3 = \lambda\beta\left(3\delta^2 + \beta^2\right),\ \kappa_4 = \lambda\left(3\delta^4 + 6\beta^2\delta^2 + \beta^4\right) (11)$$

Obviously, the cumulants enable the recovery of J-D parameters from sample moments. Press (1967), in order to avoid using higher order cumulants, imposed the restriction $\mu = 0$ and derived the following relations:

$$\hat{\beta}^4 - 2\frac{\kappa_3}{\kappa_1}\hat{\beta}^2 + \frac{3\kappa_4}{2\kappa_1}\hat{\beta} - \frac{\kappa_3^2}{2\kappa_1^2} = 0,\ \hat{\lambda} = \frac{\kappa_1}{\hat{\beta}}, \hat{\delta}^2 = \frac{\kappa_3 - \hat{\beta}^2\kappa_1}{3\kappa_1},\ \hat{\sigma}^2 = \kappa_2 - \frac{\kappa_1}{\hat{\beta}}\left(\hat{\beta}^2 + \frac{\kappa_3 - \hat{\beta}^2\kappa_1}{3\kappa_1}\right) \quad (12)$$

Press' estimates often carried the wrong-sign and were not plausible. Beckers (1981) adopted the same method as Press, however, setting β, instead of μ, to zero. Using sixth order cumulants, his cumulant equations yielded the following system:

$$\hat{\mu} = \kappa_1,\ \hat{\lambda} = \frac{25\kappa_4^3}{3\kappa_6^2},\ \hat{\delta}^2 = \frac{\kappa_6}{5\kappa_4},\ \hat{\sigma}^2 = \kappa_2 - \frac{5\kappa_4^2}{3\kappa_6} \quad (13)$$

Beckers' estimates improved those of Press, yet they were not free of anomalies. Ball and Torous (1983), using a Bernoulli, instead of a Poisson, jump process and maintaining Beckers' restriction, i.e. $\beta = 0$, derived the following cumulant equations:

$$\kappa_1 = \mu,\ \kappa_2 = \sigma^2 + \lambda\delta^2,\ \kappa_3 = 0,\ \kappa_4 = 3\delta^2\lambda(1-\lambda),\ \kappa_5 = 0,\ \kappa_6 = 15\delta^6\lambda(1-\lambda)(1-2\lambda) \quad (14)$$

Again by equating with population cumulants, they obtained estimators $\hat{\mu}$, $\hat{\lambda}$, $\hat{\sigma}^2$, and $\hat{\delta}^2$ given by:

$$\hat{\mu} = \kappa_1,\ \hat{\lambda} = \left(1 \pm \sqrt{3\kappa^*/(3\kappa^* + 100)}\right)/2,\ \hat{\sigma}^2 = \kappa_2 - \hat{\lambda}\hat{\delta}^2,\ \ \hat{\delta}^2 = \kappa_6 / \left(\kappa_4\left(5\left(1 - 2\hat{\lambda}\right)\right)\right) (15)$$

where $\kappa^* = (\kappa_6/\kappa_4)^2$. Das and Sundaram (1999) used the method of moments to estimate the J-D model. Denoting the log-price return by x_t, and assuming that the jump size J_t is distributed as $J \sim N(\beta, \delta^2)$, they computed the first moments of the J-D process. Imposing a given value for the Poisson parameter λ, they used the moments' equations to estimate the model's parameters:

$$Var[x] = E\left[\left(x - E(x)^2\right)\right] = \left(\sigma^2 + \lambda\delta^2\right) \quad (16)$$

$$skewness(x) = \frac{E\left[\left(x - E(x)\right)^3\right]}{\left[Var(x)\right]^{3/2}} = \frac{\lambda\left(\beta^3 + 3\beta\delta^2\right)}{\left(\sigma^2 + \lambda\delta^2 + \lambda\beta^2\right)^{3/2}} \quad (17)$$

[13] See, for instance, Madan and Seneta (1987), and Cont and Tankov (2004).

$$kurtosis\left(x\right)=\frac{E\left[\left(x-E\left(x\right)\right)^{4}\right]}{\left[Var\left(x\right)\right]^{2}}=3+\frac{\lambda\left(\beta^{4}+6\beta^{2}\delta^{2}+3\delta^{4}\right)}{\left(\sigma^{2}+\lambda\delta^{2}+\lambda\beta^{2}\right)^{2}} \quad (18)$$

3. The method of the characteristic function (CF): As there is a one-to-one correspondence between the CF, $\phi_X\left(u\right)=E\left[\exp\left(iuX_t\right)\right]=\int\exp\left(iuX_t\right)f\left(X_t\right)dX_t$, and the corresponding probability density, $f\left(X_t\right)$, the CF conveys the same information as the probability distribution. Often, the transition density function of a stochastic process may not be available in closed form, while the CF is readily available in closed form. Knowledge of the analytic form of the CF allows estimating the parameters of the process by the method of moments or the empirical CF procedure (ECF).[14] The method of moments computes non-central moments of any order n as $E\left[\left(X_t\right)^n\right]=\frac{1}{i^n}\frac{d^n}{du^n}\phi\left(u\right)|_{u=0}$. It also enables the application of the empirical characteristic function method (ECF). In both cases, a General Method of Moments (GMM) procedure is implemented, consisting of minimizing a distance norm between the sample and the theoretical population moments, or the sample CF and the theoretical CF. The exact method of moments consists of estimating the parameter vector which minimizes the distance $\left\| E\left(X^n\right)-\frac{1}{i^n}\frac{\partial^n\phi}{\partial u^n}|_{u=0}\right\|$.[15]

The ECF method can be described as follows. Suppose $x=\{x_1,x_2,.......,x_T\}$ is an identically independently distributed realization of the same variable X with density $f(x;\theta)$ and a distribution $F_\theta(x)$. The parameter $\theta\in R^l$ is the parameter of interest with true value θ_0. It is to be estimated from $x=\{x_1,x_2,.......,x_T\}$. Define the theoretical CF as: $\phi_\theta(u)=\int e^{iux}f(x;\theta)dx$ and its empirical counterpart (ECF) as:

$$\phi_n(u)=\int e^{iux}f_n(x)dx=\frac{1}{T}\sum_{j=1}^{T}\exp(iux_j)=\frac{1}{T}\sum_{j=1}^{T}\cos(ux_j)+i\frac{1}{T}\sum_{j=1}^{T}\sin(ux_j) \quad (19)$$

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[14] Parzen (1962), Feuerverger and Mureika (1977), Feuerverger and McDunnough (1981a and 1981b)) suggested the use of the CF to deal with the estimation of density functions. Madan and Seneta (1987) proposed a CF-based approach to estimate the J-D model. In the same vein, Bates (1996), Duffie et al. (2000), Chacko and Viceira (2003), and many other authors have proposed the use of CF for estimating affine J-D models.

[15] Note that $X^n=e^{\log\left(X^n\right)}=e^{n\log(X)}$. Therefore,

$E\left(X^n\right)=E\left(e^{Log\left(X^n\right)}\right)=E\left(e^{nLog(X)}\right)=\int_{-\infty}^{\infty}e^{nLog(X)}f\left(X\right)dX=\phi_{Log(X)}\left(n\right)$. Namely, for the log-return $x_t=\log(S_t/S_{t-1})$, $E\left(x^n\right)=E\left(e^{log\left(S_t/S_{t-1}\right)^n}\right)=E\left(e^{nx}\right)=\int_{-\infty}^{\infty}e^{nx}f\left(X\right)dX=\phi_x\left(n\right)$. It follows that the $n-th$ order moment $E\left(x^n\right)$ can be computed by replacing the transform variable u by n in the CF of $x_t=\Delta X_t=\log\left(S_t/S_{t-1}\right)$.

The ECF procedure consists of estimating θ according to the criterion:

$$\hat{\theta} = \arg\min_{\theta}(\phi_n - \phi_\theta)' W\ (\phi_n - \phi_\theta) \quad (20)$$

W is a positive semi-definite matrix. Because the minimization of the distance between the ECF (ϕ_n) and CF (ϕ_θ) over a grid of points in the Fourier domain is equivalent to matching a finite number of moments, the ECF method is in essence equivalent to the Generalized Method of Moments (GMM). Feuerverger (1990) proved that, under some regularity conditions, the resulting estimates can be made to have arbitrarily high asymptotic efficiency provided that the sample of observations is sufficiently large and the grid of points is sufficiently fine and extended. Indeed, ECF estimators have the same consistency and asymptotic efficiency as the GMM estimators. Moreover, when the number of orthogonal conditions exceeds the number of parameters to be estimated, the model is over-identified, in that more orthogonal conditions are used than needed to estimate θ. A test of over-identifying restrictions may be used. In this respect, Hansen (1982) suggested a test of whether all of the sample moments are as close to zero as would be expected if the corresponding population moments were truly zero.

4. Empirical Results of the Estimation

Based on a sample of daily prices for Brent futures prices described in Section II, alternative methods were used for estimating the J-D model (Table 2). First, assuming a Bernoulli jump process, the ML was applied unrestrictedly, and with restriction on the probability λ of a jump occurring on a trading day given by λ =0.23. Second, the method of cumulants was applied consecutively with restrictions λ =0.23, μ =0 (Press, 1967), and β =0 (Beckers, 1981), respectively. The third method was the ECF applied unrestrictedly and with restriction λ =0.23. The three methods yielded parameter estimates that were consistent with the empirical features of oil prices discussed in Section II. They showed pointedly that the dynamics of the oil price process were influenced by both diffusion and jump components; however the jump component was dominant. Besides having high intensity, the jump component had a much higher variance than the diffusion component. The high variance of the jump component illustrated the presence of jumps of large magnitude and was in conformity with the excess kurtosis in the empirical distribution of oil price returns. The mean of the jump size tended to be negative, in conformity with the negative skewness of the empirical distribution. This was due to the fact that crude oil prices were not monotonic; they leapt forward, than retreated back in smaller movements before taking a new jump. The drift of the diffusion component was high, in conformity with the observed upward trend in crude oil prices; it illustrated the presence of a force that kept pushing oil prices upward and was able to outweigh the negative mean of the jump component.

Table 2. Jump-Diffusion Model: Parameter Estimates

Methods	Drift μ	Variance σ^2	Intensity λ	Mean β	Variance δ^2
Bernoulli process					
Maximum Likelihood	0.23	4.46	0.59	-1.12	4.47
	(t=3.22)	(t=20.25)	(t=1.89)	(t=-4.27)	(t=17.12)
Maximum Likelihood	0.27	3.34	0.23	-0.68	7.98
1/	(t=3.08)	(t=14.49)		(t=-1.93)	(t=6.25)

Cumulants 1/	0.32	1.81	0.23	-0.85	2.78
Press (1967) 2/	0	6.54	0.10	1.11	-13.88
Beckers (1981) 3/	0.12	3.34	0.22	0	8.62
ECF 4/	0.57	0.54 (t=1.11)	4.37	-0.10	0.96 (t=5.57)
	(t=7.96)	3.45	(t=3.49)	(t=-6.25)	6.97
ECF 1/ 4/	0.27	(t=127)	0.23	-0.52	(t=31.35)
	(t=16.1)			(t=-3.75)	

1/ Restriction on $\lambda = 0.23$, computed from the data sample as the frequency of a jump in the crude oil price exceeding ± 3 percent. 2/ Restriction on μ =0. 3/ Restriction on β =0. 4/ The grid for u consists of twenty points: $0.1, 0.2, 0.3, ..., 1.9, 2.0$.

Assuming a Bernoulli jump process, the ML estimates were highly significant and stable. The drift of the diffusion component, estimated at $\hat{\mu}$ =0.23, was very high and significant, showing that oil prices were constantly under upward pressure. The variances of the diffusion and jump components were high and significant, $\hat{\sigma}^2$ =4.46 and $\hat{\delta}^2$ =4.47, respectively. The variance of the jump component became more important than that of the diffusion component when the jump intensity was restricted to λ =0.23. The probability of a jump in the unrestricted case, computed at $\hat{\lambda}$ =0.59, was high and borderline significant. The mean of the jump component, estimated at $\hat{\beta}$ =-1.12, was negative and consistent with the negative skewness observed in the data. Oil prices tended to make large moves upward, then started to retreat through a sequence of smaller and frequent negative jumps, until they were shocked again, making new jumps forward. Yet, the significance of the drift of the diffusion process was such that the smaller negative jumps could not outweigh the strong momentum that kept pushing oil prices upward.

The method of cumulants was applied under alternative restrictions. The restriction λ =0.23 yielded results that were similar to the ML under the same restriction. The drift of the diffusion component, estimated at $\hat{\mu}$ =0.32, was very high, showing that oil prices were constantly under pressure to move upward. The variances of the diffusion and jump components, were estimated at $\hat{\sigma}^2$ =1.81 and $\hat{\delta}^2$ =2.78, respectively, indicating that the jump component tended to dominate the dynamics of the oil price process. The mean of the jump component, estimated at $\hat{\beta}$ =-0.85, was negative and consistent with the negative skewness in oil price returns. Application of the Press (1967) method, with the restriction μ =0, yielded implausible results for the variance of the jump component, namely $\hat{\delta}^2$ =-13.88. Such an anomaly was not unexpected in the case of Press' method, indicating that the restriction μ =0, could not be borne by the data, and was in sharp contrast with the strong upward trend in oil prices. In contrast, Beckers' method, with the restriction β =0, yielded results which were highly plausible. The drift component of the diffusion, estimated at $\hat{\mu}$ =0.12, was smaller than, say, in the ML case, since β =0 implied less influence for the drift of the diffusion, compared to the case when β was negative, to maintain an upward trend in oil prices; it was, however, close to the drift of the AR2 (Table 1) and the actual mean of oil price returns (Figure 2). The variances of the diffusion and jump components were high, $\hat{\sigma}^2$ =3.34 and $\hat{\delta}^2$ =8.62, respectively. The variance of the jump component, however, dominated that of the diffusion component. Noticeably, the jump intensity, estimated at $\hat{\lambda}$ =0.22, was quite close to the frequency of jumps in oil prices exceeding $\pm$ 3 percent, computed from the data set.

The ECF was applied unrestrictedly and with restriction λ =0.23. The drift of the diffusion component, estimated at $\hat{\mu}$ =0.57, was very high and significant. The variance of the diffusion component, $\hat{\sigma}^2$ =0.54, was not significant, and was dominated by the variance of the jump component, $\hat{\delta}^2$ =0.96, which was significant. The intensity of the jump process, estimated at $\hat{\lambda}$ = 4.37, was high and significant, indicating that the oil price process was characterized by frequent jumps. The mean of the jump component, $\hat{\beta}$ =-0.1, was negative, significant, and consistent with skewness in oil price returns. The ECF, applied with the restriction λ =0.23, yielded results which were similar to those of the ML using the same restriction. The drift $\hat{\mu}$ =0.27 was positive and significant; the variance of the diffusion, $\hat{\sigma}^2$ =3.34, was significant; however, it was dominated by the variance of the jump component, $\hat{\delta}^2$ =6.97, indicating that the jump process played a more important role in oil price dynamics in relation to the diffusion process. The mean of the jump component, $\hat{\beta}$ =-0.52, was negative, significant and consistent with skewness observed in the data.

In sum, parameter estimates from the three methods were fully concordant with the data. They established that the oil price process was dominated by the jump process, with large discontinuities occurring at high intensity, meaning that oil markets were permanently out-of-equilibrium during the sample period. The negative mean of the jump component could be seen as smaller downward adjustment in world crude oil demand following a large upward jump in oil prices. However, the downward adjustment in demand was short-lived; the drift component of the diffusion process was very high for daily data, indicating that oil demand was pushed up by a strong income effect; consequently, oil prices were under a constant pressure to move upward. These results can be explained given global elasticities of demand and supply for crude oil. World demand was highly elastic with respect to world income, and highly inelastic with regard to oil prices. Crude supply has been rigid, showing little sensitivity to prices. As world real GDP expanded at 4–5 percent per year during the period under study, it caused world oil demand to expand at similar rate, creating an excess demand for oil. Given the short-term inelasticity of demand and supply with respect to prices, any small excess demand for oil would cause large variation in prices. In turn, large price increases would have small negative effect on oil demand. The negative price effect, however, would be quickly dominated by a positive income effect.

IV. Crude Oil Price as a Variance-Gamma Levy Process

The J-D model has essentially two limitations. First, it does not capture the notions of time-varying and stochastic volatility. In particular, stochastic volatility is found to have a key role in explaining skewness and leptokurtosis in financial time-series and the skew in market implied volatilities. In this respect, skewed distribution can arise either because of correlations between asset prices and volatility shocks, or because of nonzero average jumps. Similarly, excess kurtosis can arise either from volatile volatility or from a substantial jump component. Second, the J-D model is fit to model finite large jumps, and cannot capture infinite small jumps which are similar to small jumps in the diffusion process. With a view to capturing the notion of stochastic volatility and modeling small and frequent jumps, while simplifying computational costs, many researchers (e.g., Carr et. al (2002, 2003), Carr and Wu (2004), Cont and Tankov (2004)) have proposed the use of Levy processes for modeling asset prices. Accordingly, oil prices are modeled in this section as a Levy process.[16] More specifically, oil price returns are assumed to follow a Levy process with a variance-gamma distribution. This type of model has a simple CF and is easier to estimate.

1. Definition of the Variance-Gamma process

A variance-gamma (VG) process is defined as a Brownian motion with drift α and volatility σ, i.e. $\alpha t + \sigma B_t$, where B_t is an ordinary Brownian motion, time-changed by a gamma process. More precisely, let $G = \{G_t, t \geq 0\}$ be a gamma process with mean $a = 1/\upsilon > 0$ and variance $b = 1/\upsilon > 0$.[17] Let $B = \{B_t, t \geq 0\}$ denote a Brownian motion, and let $\sigma > 0$ and $\alpha \in R$; then the VG process $X^{(VG)} = \{X_t^{(VG)}, t \geq 0\}$, with parameters $\sigma > 0$, $\upsilon > 0$ and α, can be defined as $X_t^{(VG)} = \alpha G_t + \sigma B_{G_t}$. The CF is given by:

$$\phi_{VG}(u;\sigma,\upsilon,\alpha) = E[\exp(iuX_t^{(VG)})] = (1 - iu\alpha\upsilon + \frac{1}{2}\sigma^2\upsilon u^2)^{-\frac{t}{\upsilon}} \quad (20)$$

The two additional parameters in the VG distribution, which are the drift of the Brownian motion, α, and the volatility of the time change, υ, provide control over skewness and kurtosis, respectively. Namely, when $\alpha < 0$, the distribution is negatively skewed, and vice versa. Moreover, larger values of υ indicate frequent jumps and contribute to fatter tails. The moments of the log-price returns under $VG(\sigma,\upsilon,\alpha)$ are: the mean $= \alpha$; the variance

[16] A Levy process (LP) $(X_t)_{t\geq 0}$ has a value $X_0 = 0$ at $t = 0$ and is characterized by independent and stationary increments, and stochastic continuity, i.e., discontinuity occurs at random times. The CF of a LP is given by the Levy-Khintchine formula:

$$\phi(u) = E[e^{iuX_1}] = \exp\left(i\alpha u - \frac{\sigma^2}{2}u^2 + \int_{R\setminus\{0\}} [e^{iux} - 1 - iux 1_{|x|<1}(x)]\nu(dx)\right), u \in R, t \geq 0.$$ Where $\alpha \in R$ is the drift parameter, $\sigma^2 \geq 0$ is the volatility parameter, and ν is a Levy measure on $R\setminus\{0\}$, which measures jumps of different sizes. A LP is characterized by its triplet (α, σ^2, ν).

[17] The probability density of the Gamma process with mean rate t and variance υt is well known: $f(x) = x^{\frac{1}{\upsilon}-1} e^{-\frac{x}{\upsilon}} / \upsilon^{\frac{t}{\upsilon}} \Gamma(\frac{t}{\upsilon})$. Its Laplace transform is $E[\exp(-uG_t^{\upsilon})] = (1 + u\upsilon)^{-\frac{t}{\upsilon}}$.

The result is that the VG process has a simple CF $\phi_{VG}(u) = 1/(1 - i\alpha\upsilon u + \frac{\sigma^2\upsilon}{2}u^2)^{\frac{t}{\upsilon}}$.

$=\sigma^2+\upsilon\alpha^2$; skewness $=\dfrac{\alpha\upsilon(3\sigma^2+2\upsilon\alpha^2)}{(\sigma^2+\upsilon\alpha^2)^{3/2}}$; and kurtosis $=3(1+2\upsilon-\upsilon\sigma^4(\sigma^2+\upsilon\alpha^2)^{-2})$.
Clearly, skewness is influenced by α, and kurtosis by υ.

2. Estimation of the Variance-Gamma process

Let the crude oil price S_t be modeled as $S_t = S_0\exp[\mu t + X_t]$ where X_t is a VG process. The log price return is $x_t = \Delta\log S_t = \mu + X_t - X_{t-1}$. Because of infinite divisibility of the VG process, the CF of the log price return is:

$$E[\exp(iux_t)] = E\left[\exp(iu(\mu+X_1)\right] = \exp(iu\mu)E\left[\exp(iuX_1)\right]$$
$$= \exp(iu\mu)(1-i\alpha\upsilon u+\frac{1}{2}\sigma^2u^2\upsilon)^{-\frac{1}{\upsilon}} \qquad (21)$$

Using the data described in Section II, the parameters were estimated using the ECF approach (Table 3).[18] The estimated parameters of the VG process were stable and statistically significant, and corroborated the findings for the J-D process. Namely, when modeled as a VG process, crude oil prices exhibited a high drift coefficient, high volatility, frequent and large jumps, and skewness. The drift coefficient, estimated at $\hat{\mu}=0.09$, asserted the presence of a strong upward pulling force which kept reigniting oil prices. The volatility parameter, $\hat{\sigma}=1.66$, was high and significant, showing that oil markets were facing high uncertainties regarding future movements in prices. The parameter υ, which controls for tail fatness, estimated at $\hat{\upsilon}=0.80$, was high and significant, implying the presence of frequent and large jumps in oil prices. The parameter α, which controls for skewness, estimated at $\hat{\alpha}=-0.36$, was negative and significant, showing that the VG distribution was left-skewed. More specifically, in response to large positive jumps in oil price, there seemed to be a cooling off period during which world crude oil demand might slowdown, causing small and frequent negative jumps in prices. However, the income demand elasticity was much higher and more significant than the price elasticity; thus faster world economy growth kept pushing world oil demand upward.

Table 3. Parameter Estimates of the VG process 1/

Drift μ	Drift α	Volatility σ	Variance of VG υ
0.09 (t=3.52)	-0.146 (t= -2.54)	1.66 (t= 80.4)	0.804 (t=52.3)

1/ Using ECF method.

V. Option Pricing Using Characteristic Functions

In this section we present some basic elements of option pricing with a view to paving the way for inferring oil price density forecast from options' prices. The estimation of the risk-

[18] The VG process has also been estimated using R and the package ghyp developed by Wolfgang Breymann and David Luthi: www.r-project.org. The estimates were very close to the ones reported here.

neutral distribution is known as the inverse problem in option pricing models. While the pricing problem is concerned with computing option values given model's parameters, the inverse problem consists of backing out the parameters describing risk-neutral dynamics from observed prices. The computation of a risk-neutral distribution could be seen as estimating market's expectations for future prices, paying attention not only to the mean, which is observed directly from futures prices, but also to skewness (direction of trends) and kurtosis (risk for large fluctuations or crash); in contrast, estimation of statistical distribution from realized data could be seen as the actual distribution of historical prices.

Option Value in the Asset Price Space: Under martingale pricing, the value of an option, denoted by $f(S,t)$, is a convolution of a discounted pay-off function with the state price density. For a given final condition (pay-off): $f(S,T) = g(S)$ for all S, the option value is:

$$f(S_t,t) = E_t^Q[e^{-r(T-t)}g(S_T) \mid S_t = S] = \int_0^\infty e^{-r(T-t)}g(S_T)p(S_T \mid S_t)dS_T \quad (22)$$

Where Q is the risk-neutral measure. The conditional expectation is computed with respect to a risk-neutral transition probability density $p(S_T \mid S_t)$. However, for many stochastic processes involving stochastic volatility, jumps, or Levy type processes, transition densities are often complicated and may not be readily available in closed form. In contrast, the CF of the underlying stochastic process may be readily available in closed form. It is defined as:

$$\phi(u,t) = \int_0^\infty e^{iuS_T} p(S_T \mid S_t)dS_T \quad (23)$$

Where u is the transform variable.

Option Value in the Fourier Space: knowledge of the CF enables the computation of option prices in the Fourier space according to two alternative methods. The first method, proposed by Heston (1993), relies on a numerical inversion of the CF. However, noting the singularity of Heston's formula at $u = 0$, Carr and Madan (1999) proposed, instead, a numerical inversion of the Fourier transform of the option value. More specifically, define the Fourier transform of the option value as:

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{iuS} f(S)dS \quad (24)$$

If $\hat{f}(u)$ can be explicitly expressed in terms of $\phi(u)$ as:

$$\hat{f}(u) = \hat{f}_\phi(\phi(u)) \quad (25)$$

a fast Fourier transform (FFT) inversion of $\hat{f}_\phi(\phi(u))$ would then compute the option value from its transform as:

$$f(S,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuS}\hat{f}(u)du \quad (26)$$

Let $s_t = \ln S_t$ be the log-price; $k = \ln(K)$ the log strike price; $C_T(k)$ = value of a $T-$maturity call option with strike K; and $c_T(k) \equiv \exp(ak)C_T(k)$ for $a > 0$, the damped option price. The CF of $s_t = \ln S_t$ under the risk-neutral measure is given by $\phi(u,t) = \int_{-\infty}^{\infty} e^{ius_T} p(s_T \mid s_t)ds_T$. Let

$\psi_T(u) = \int_{-\infty}^{\infty} e^{iuk} c_T(k)dk$ be the Fourier transform of $c_T(k)$. Carr and Madan (1999) showed that $\psi_T(u)$ can be expressed in terms of $\phi_T(u)$ as:

$$\psi_T(u) = \frac{e^{-rT}\phi_T(u-(a+1)i)}{a^2+a-u^2+i(2a+1)u} \quad (27)$$

Knowledge of the CF $\phi_T(u)$, which is the CF of the log of the asset price under the risk neutral-measure, implies knowledge of the Fourier transform of the value of the option $\psi_T(u)$. The option price can therefore be computed via Fourier inversion as:

$$C_T(k) = \frac{\exp(-ak)}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\psi(u)du = \frac{\exp(-ak)}{\pi}\int_{0}^{\infty} e^{-iuk}\psi(u)du \quad (28)$$

Fourier or Laplace-based methods for pricing options were further expanded. For instance, noting that an option price has two components, which are the intrinsic value defined as $(S_T - K)^+$ and the time value, and that the latter component is square integrable for out-of-money options, Cont and Tankov (2004) proposed to compute the time value and derive the option value by adding the intrinsic value. Indeed, denoting the time value of an option by $z_T(k)$, its Fourier transform is:

$$\zeta_T(u) = \int_{-\infty}^{\infty} e^{iuk} z_T(k)dk \quad (29)$$

Cont and Tankov (2004) established that:

$$\zeta_T(u) = \frac{e^{-rT}\phi_T(u-i) - e^{iurT}}{iu(1+iu)} \quad (30)$$

Option prices can be found by inverting the Fourier transform:

$$z_T(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\zeta_T(u)du \quad (31)$$

Most notable extension was due to Lewis (2001) who showed that the option value can be expressed as a convolution of generalized Fourier transforms:[19]

$$C(S_T, K) = \frac{e^{-rT}}{2\pi}\int_{i\varpi-\infty}^{i\varpi+\infty} e^{-izX}\phi(-z)\hat{g}(z)dz \quad (32)$$

where $\ln S_t = \ln S_0 + X_t + rt$, $\hat{g}(z) = \int_{-\infty}^{\infty} \exp(izs)g(s)ds$, $z = u + iw$ is a complex number, $s_t = \ln S_t$, and $(i\varpi - \infty, i\varpi + \infty)$ is an integration line parallel to the real axis, with ϖ

[19] This formula is based on Parseval's identity: $\int_{-\infty}^{\infty} g(S_T)p(S_T)dS_T = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(z)\phi^*(z)dz$, where $\hat{g}(z)$ is the Fourier transform of $g(S_T)$ and $\phi^*(z)$ is the conjugate of $\phi(z)$, i.e., $\phi^*(z) = \phi(-z)$.

satisfying integration conditions (See Cont and Tankov, 2004). Being expressed as a complex-valued integral, the option value can thus be computed using residue calculus.[20] Option pricing requires knowledge of the risk-neutral process and the CF associated with this process. Under the risk neutral process, money market account discounted asset prices are martingales and it follows that the mean rate of return on the asset under this probability measure is the continuous compounded risk-free interest rate r. If the asset price S_t is modeled as $S_t = S_0 \exp[rt + X_t]$ where X_t is an LP, to obtain a risk-neutral process verifying the martingale property, define:

$$S(t) = S(0)\frac{\exp(rt + X(t))}{E[\exp(X(t)]} = S(0)\frac{\exp(rt)\exp(X(t))}{E[\exp(X(t)]} \quad (33)$$

then $E\left(S(t)/\exp(rt)\right) = S(0)$. The resulting risk-neutral process for the log price is: $\log S(t) = (\log S(0) + rt - \log E[\exp(X(t)]) + X(t))$. The CF of the log price is:

$$E[\exp(iu\log(S(t)))] = \exp(iu((\log S(0) + rt - \log E[\exp(X(t)])E[\exp(iuX(t))] \quad (34)$$

For the VG model, the resulting risk-neutral process for the asset price is:

$$S_t = S_0 \exp[rt + X_t(\sigma,\alpha,\upsilon) + \omega t],\ t > 0 \quad (35)$$

where, by setting $\omega = (\frac{1}{\upsilon})\ln(1 - \alpha\upsilon - \frac{1}{2}\sigma^2\upsilon)$, Madan et al. (1998) showed that the CF for log of S_t is:

$$\phi_t(u) = \exp[\ln(S_0) + (r + \omega)t](1 - i\alpha\upsilon u + \frac{1}{2}\sigma^2 u^2\upsilon)^{-\frac{t}{\upsilon}} \quad (36)$$

To obtain option prices, one can analytically invert $\phi(u)$ to get the density function and then integrate the density function against the option payoff as in Heston (1993). Alternatively, the Fourier transform of the option value can be numerically inverted using FFT as in Carr and Madan (1999) and other Fourier-Laplace methods. The Fourier inversion can be approximated discretely via an N-point sum with a grid spacing of Δ in the Fourier domain. The inversion integral can be approximated using an integration rule, such as Simpson's or the trapezoidal rule, as:

$$\int_0^{\infty} e^{-iku}\psi(u)du \approx \sum_{j=0}^{N-1} e^{-ik(\frac{2\pi}{N}j)u_j}\tilde{\psi}_j\Delta \quad (37)$$

[20] Lewis showed that the Fourier transform of a call payoff is given by $\hat{g}(z) = \int_{-\infty}^{\infty} e^{izx}\left(e^x - K\right)^+ = -\frac{K^{iz+1}}{\left(z^2 - iz\right)}$, $\operatorname{Im} z > 1$. For a put option, the payoff transform is $\hat{g}(z) = \int_{-\infty}^{\infty} e^{izx}\left(K - e^x\right)^+ = -\frac{K^{iz+1}}{\left(z^2 - iz\right)}$, $\operatorname{Im} z < 0$. Accordingly, a European call can be priced as follows: $C(S_t,\tau;K) = -\frac{Ke^{-r\tau}}{2\pi}\int_{i\varpi-\infty}^{i\varpi+\infty} e^{-izk}\phi(-z)\frac{K^{iz+1}}{\left(z^2 - iz\right)}dz$, where $k = \log\left(S_0/K\right)$.

The points u_j are equidistant with grid spacing Δ, $u_j = j\Delta$. The value of Δ should be sufficiently small to approximate the integral well enough, while the value of $N\Delta$ should be large enough to assume the CF is equal to zero for $u > \bar{u} = N\Delta$. In general, the values $\tilde{\psi}_j$ are set equal to $\tilde{\psi}_j = \psi(u_j)w_j$, where w_j are the weights of the integration rule.

VI. Density Forecast of Crude Oil Prices: The Inverse Problem

An application of the above analysis to crude oil options is undertaken in this section with the objective of estimating, from observed options' market values, density forecast for crude oil prices at a given maturity date. Assuming a VG distribution for the log price, the inverse problem can be stated as finding the parameters $\theta = (\alpha, \sigma^2, \upsilon)$ by minimizing the quadratic pricing error:

$$\hat{\theta} = \arg\min_{\theta} \frac{1}{M}\sum_{j=1}^{M}\left(C_j^*(T,k_j) - C_j(T,k_j)\right)^2, \quad j = 1,2,....,M \quad (38)$$

under the put-call parity constraint: $S_0 + P_j(T,k_j) - C_j^*(T,k_j) = k_j e^{-rT}$

where $C_j^*(T,k_j)$ denotes the call option computed from the VG distribution, $C_j(T,k_j)$ and $P_j(T,k_j)$ denote the observed prices of call and put options for maturity T and strikes k_j, respectively and M denotes the number of traded options (or strikes). $C_j^*(T,k_j)$ is given by FFT; namely $C_j^*(T,k_j) = \dfrac{\exp(-ak_j)}{\pi}\int_0^{\infty} e^{-iuk_j}\psi_T(u)du$.

The addition of the put-call parity condition brings extra-sample information which helps to regularize the estimation problem.[21] Taking into account the put-call parity constraint and choosing a penalty parameter $h > 0$, the minimization problem becomes:

$$\hat{\theta} = \arg\min_{\theta} \frac{1}{M}\sum_{j=1}^{M}\left(\left(C_j^*(T,k_j) - C_j(T,k_j)\right)^2 + h\left(S_0 + P_j(T,k_j) - C_j^*(T,k_j) - K_j e^{-rT}\right)^2\right) \quad (39)$$

The estimation of the implied risk-neutral distribution from option prices is a deconvolution problem. Madan et al. (1998) applied maximum likelihood method to the density function to calibrate a VG process based on option prices. In this section, we remain consistent with deconvolution methods based on characteristic functions as these functions were found to satisfy the same differential equations or least squares problems as the corresponding option prices. As in Section IV, the estimation method relies principally on the empirical characteristic function. The minimization problem is restated as:

[21] Cont and Tankov (2004) argued that the inverse problem could be an ill-posed problem and proposed the use of relative entropy, which is the Kullback-Leibler distance for measuring the proximity of two equivalent probability measures, as a regularization method with the prior distribution estimated from the statistical data via the maximum likelihood method. This regularization will enable the finding of a unique martingale measure.

$$\hat{\theta} = \arg\min_{\theta} \frac{1}{N}\sum_{i=1}^{N}\left(\begin{array}{l}\left(\psi_T(u_i) - \frac{1}{M}\sum_{j=1}^{M} e^{iu_i k_j} e^{ak_j} C_j(T,k_j)\right)^2 \\ +h\left(S_0 \frac{1}{M}\sum_{j=1}^{M} e^{iu_i k_j} e^{\alpha k_j} + \frac{1}{M}\sum_{j=1}^{M} e^{iu_i k_j} e^{ak_j} P_j\left(T,k_j\right) - \psi_T(u_i) - \frac{e^{-rT}}{M}\sum_{j=1}^{M} e^{iu_i k_j} e^{ak_j} K_j\right)^2\end{array}\right) \quad (40)$$

Where N is the number of steps in the Fourier grid. Note that the dampening factor in Carr and Madan method, namely e^{ak_j}, would not be required if one uses Fourier transform of the time value of the option or Lewis method. The Fourier transforms of e^{ak_j}, $e^{ak_j}C_j(T,k_j)$, $e^{ak_j}P_j(T,k_j)$, and $e^{ak_j}K_j$ were computed using FFT.[22]

To check on the robustness of the estimated parameters, an alternative calibration method has been used. Let $V = \left(C_1,....,C_M,P_1,.....,P_M\right)'$ be a vector of observed call and put option prices, let D be a matrix of payoffs with dimensions $\left(2M, N_S\right)$ where $N_s \geq M$ is the number of states. Observed option prices V are related to the empirical risk-neutral distribution, q, as follows:[23]

$$V = e^{-rT}.D.q \quad (41)$$

The risk-neutral distribution is computed using Tikhonov regularization method described in Engle et al (1996) as:[24]

$$\hat{q} = e^{rT}.\left(D'D + h.I\right)^{-1}.D'.V \qquad (42)$$

Where $h > 0$ is a penalty parameter. By taking the Fourier transform of $\hat{q}$, the parameters of the distribution are estimated using the empirical characteristic function method.

The inverse problem was applied for the VG model only for space limitation. The same methodology applies identically to the J-D model.[25] The observed data set was for July 21,

[22] Packages for computing FFT can be downloaded from http://www.fftw.org/.

[23] This equation can be restated with a view to using the call-put parity condition. Let $D_C\left(M,N_S\right)$ and $D_P\left(M,N_S\right)$ be the payoff matrices associated with call and put options respectively; let also , $V_C = \left(C_1,....,C_M\right)'$ and $V_P = \left(P_1,....,P_M\right)'$ be observed call and put option prices, V_K be a vector of strikes, and $V_1 = (1,......,1)'$ be the unit vector, then: $V_C = e^{-rT}.D_C.q$ subject to: $S_0 V_1 + V_P - e^{-rT}.D_C.q = e^{-rT}V_K$

[24] The computation of $\hat{q}$ was carried out using the Matlab package by C. Hansen (1998): Regularization Tools A Matlab Package for Analysis and Solution of Discrete Ill-Posed Problems.

[25] The risk-neutral CF for any asset price model is given: $E[\exp(iu\log(S(t)))] = \exp(iu((\log S(0) + rt - \log E[\exp(X(t)])E[\exp(iuX(t))]$. For the J-D model,

2006; it consisted of call and put futures options contracts maturing end-September 2006; the risk-free interest rate, taken here to be the three-month US Treasury bill rate, was equal to 4.965; and the crude futures price, was equal to US$74.43/bl. The constrained minimization yielded the following triplet for the risk-neutral distribution: $\hat{\sigma}^2 = 1.72$, $\hat{\upsilon} = 1.12$, $\hat{\alpha} = 0.37$ which described market's expectations on July 21, 2006 regarding futures prices for end-September 2006. Clearly, market participants did not anticipate any short-term change in the underlying fundamentals characterizing oil markets. They expected oil prices to remain highly volatile ($\hat{\sigma}^2 = 1.72$) and dominated by a jump process ($\hat{\upsilon} = 1.12$). They also expected oil prices to remain under pressure, as they assigned higher probabilities for oil prices to rise above the futures price level than to fall below this level. This was shown by a right-skewed risk-neutral distribution ($\hat{\alpha} = 0.37$).

VII. Conclusions

Despite the importance of oil prices, little is known about their underlying stochastic process. Our main findings are that these dynamics are dominated by frequent jumps, causing oil markets to be constantly out-of-equilibrium. While oil prices attempted to retreat following major upward jumps, there was a strong positive drift which kept pushing these prices upward. Volatility was high, making oil prices very sensitive to small shocks and to news. The findings for both the J-D and VG specification were fully consistent with the underlying fundamentals of oil markets and world economy. More specifically, faster world economic growth during the sample period and highly expansionary monetary policies caused demand for crude oil to expand at similar pace. Given price inelastic oil demand and supply, any small excess demand (supply) would require a large price increase (decrease) to clear oil markets; hence, the observed high intensity of jumps and the strong stimulus for oil prices to rise.

When modeled as a jump-diffusion (J-D) process, oil price dynamics were dominated by the discontinuous Poisson jump component compared to the continuous Gaussian diffusion component, showing that oil markets were constantly out-of-equilibrium during the sample period and were sensitive to demand and supply shocks and to news. While the variance of the diffusion component was high and significant, it was surpassed by a still higher and significant variance of the jump component. Both variances, together, illustrated the high volatility of the oil markets. The drift of the diffusion component was, however, very high and significant, indicating that oil prices were strongly influenced by an upward trend. The mean of the jump component was negative; more specifically, sharp upward jumps in oil prices had a temporary restraining effect on oil demand and were followed by a short-lived

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the resulting risk-neutral process for the asset price is:

$$\phi_T(u) = \exp[\ln(S_0) + (r+\omega)T\,]\exp\left(T\left[-\frac{\sigma^2 u^2}{2} + i\,\mu u + \lambda\left(\exp\left(i\,\beta u - \frac{\delta^2 u^2}{2}\right) - 1\right)\right]\right)$$

where $\omega = \left[\frac{\sigma^2}{2} + \mu + \lambda\left(\exp\left(\beta + \frac{\delta^2}{2}\right) - 1\right)\right]$.

sequence of price declines. The mean of the jump component was, however, outweighed by the drift of the diffusion component, which kept prices on a rising trajectory.

Oil prices were also modeled as a Levy process (LP) with a variance-gamma (VG) distribution. The findings were similar to the J-D model. The drift component was positive and highly significant, establishing that oil prices were constantly pulled by an upward trend. The variance of the VG distribution was significant and high. The parameter controlling for the jump process was high and significant, indicating that oil prices were largely dominated by the jump component and oil markets were constantly out-of-equilibrium. The skewness of the VG distribution was negative, indicating that large upward moves in oil prices triggered a temporary depressing effect on world oil demand, translating into a temporary sequence of small negative jumps in oil prices. However, the upward momentum outweighed the small negative jumps. Turning to market expectations, the implied risk-neutral distribution from call and put option prices, assuming a VG process, showed that market participants held higher probabilities for oil prices to rise than to fall above the futures price, and expected oil prices to remain volatile and dominated by a jump process.

Our findings are relevant for policymakers and industry analysts. They establish the nature of the stochastic process underlying oil prices and the importance of components driving this process. An explanation of the process parameter estimates in terms of the underlying fundamentals for the oil markets are offered in order to comprehend the economics underpinning the observed oil prices dynamics. Namely, a change in the process parameters would require a change in the underlying fundamentals. Our alternative modeling approaches are highly relevant for forecasting, risk management, derivatives pricing, and gauging market's sentiment. Our findings could be helpful for monitoring oil markets and developing policies for stabilizing oil markets.

Annex: Method of Cumulants of Probability Distributions

Suppose that X is a real random variable whose real moment generating function is defined as: $M(u) = E\left(e^{uX}\right) = \int_{-\infty}^{\infty} e^{uX} f(X)\,dX$, where $f(X)$ is the probability density of X. Just as the moment generating function M of X generates its moments, the logarithm of M generates a sequence of numbers called cumulants. The cumulants κ_n of the probability density of X are given by: $M(u) = E\left(e^{uX}\right) = 1 + \sum_{n=1}^{\infty} \frac{m_n u^n}{n!} = \exp\left(\sum_{n=1}^{\infty} \frac{\kappa_n u^n}{n!}\right)$

Where $m_n = E\left(X^n\right)$ is the moment of order n of X. The left-hand side of this equation is the moment-generating function, so $\kappa_n/n!$ is the nth coefficient in the power series representation of the logarithm of the moment-generating function. The logarithm of the moment-generating

function is therefore called the cumulant-generating function, written as:[26]
$\log\left(M\left(u\right)\right)=\sum_{n=0}^{\infty}\frac{\kappa_n u^n}{n!}$. The method of cumulants attempts to recover a probability distribution from its sequence of cumulants. In some cases no solution exists; in other cases a unique solution, or more than one solution, exists. The relationship between moments and cumulants is of paramount importance in the estimation of the unknown parameters of the density function. First, consider moments about 0, which can be written as $m_j = E\left(X^j\right)$, $j = 0,1,2...$ The cumulant/moment theorem says that if X is a random variable with n moments m_1, m_2 ,......, m_n , then X has n cumulants κ_1, κ_2 ,....., κ_n , and the cumulants are related to the moments by the following recursion formula:[27] $\kappa_n = m_n - \sum_{j=1}^{n-1}\binom{n-1}{j-1}\kappa_n m_{n-j}$

Note that $m_0 = 1$. By carrying the recursion formula, the relation between raw moments and cumulants can be stated as:

$$m_1 = \kappa_1$$
$$m_2 = \kappa_2 + m_1\kappa_1$$
$$m_3 = \kappa_3 + 2m_1\kappa_2 + m_2\kappa_1$$
$$m_4 = \kappa_4 + 3m_1\kappa_3 + 3m_2\kappa_2 + m_3\kappa_1$$

For central moments, defined by $\overline{m}_j = E\left(\left(X - E\left(X\right)\right)^j\right)$, the first moment $\overline{m}_1$ is zero; the relationship between moments and cumulants simplifies to:

$$\overline{m}_1 = \kappa_1 = 0$$
$$\overline{m}_2 = \kappa_2$$
$$\overline{m}_3 = \kappa_3$$
$$\overline{m}_4 = \kappa_4 + 3\overline{m}_2\kappa_2$$

The first cumulant is simply the expected value; the second and third cumulants are respectively the second and third central moments (the second central moment is the variance); but the higher cumulants are neither moments nor central moments, but rather

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[26] The cumulants are also equivalently defined in terms of the characteristic function, which is the Fourier transform of the probability density function: $\phi\left(u\right)= E\left(e^{iuX}\right)= \int_{-\infty}^{\infty} e^{iuX} f\left(X\right)dX$. The cumulants κ_n are then defined as: $\ln\phi\left(u\right)= \sum_{n=1}^{\infty}\kappa_n\frac{\left(iu\right)^n}{n!}$

[27] This recursion formula is the Faa di Bruno’s formula, equivalently written as:

$$m_{r+1} = \sum_{j=0}^{r}\binom{r}{j}m_j\kappa_{r+1-j} \text{ for } r = 0......, n-1$$

more complicated polynomial functions of the moments. The *n*th moment m_n is an *n*th-degree polynomial in the first *n* cumulants. Of particular interest is the fourth-order cumulant, called kurtosis, which can be expressed as $kurt(X) = E(X^4) - 3(E(X^2))^2$. Kurtosis can be considered as a measure of the non-Gaussianity of X. For a Gaussian random variable, kurtosis is zero; it is typically positive for distributions with heavy tails and a peak at zero, and negative for flatter densities with lighter tails.

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